An illustrated introduction to Euler and Fitting factorizations and Anderson graphs for classic magic matrices

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Abstract

We build on results [8] about Euler factorizations of magic matrices presented at the LINSTAT'2008 Conference in celebration of Tadeusz Caliński's 80th Birthday. Our classic magic matrices are $n \times n$ with entries 0, 1, ..., $n^2 - 1$ in some order. These matrices are fully-magic in that the numbers in all rows, columns, and the two main diagonals all add up to the same *magic sum*. The 4×4 classic magic matrix $\mathbf{M}$ has *Euler factorization* $\mathbf{M} = 4\mathbf{L}_1 + \mathbf{L}_2$, where the first *Euler component matrix* $\mathbf{L}_1 = \left[\frac{1}{4}\mathbf{M}\right]$ is the 4×4 matrix with entries which are the integer parts of entries in $\frac{1}{4}\mathbf{M}$. We also build on seminal results [5] by Friedrich Fitting (1862–1945) and his son Hans Fitting (1906–1938) about the factorization $\mathbf{M} = 8\mathbf{B}_1 + 4\mathbf{B}_2 + 2\mathbf{B}_3 + \mathbf{B}_4$, where the binary *Fitting component matrices* $\mathbf{B}_1 = \left[\frac{1}{8}\mathbf{M}\right]$, $\mathbf{B}_2 = \left[\frac{1}{4}\mathbf{M} - 2\mathbf{B}_1\right]$ and $\mathbf{B}_3 = \left[\frac{1}{2}\mathbf{M} - 4\mathbf{B}_1 - 2\mathbf{B}_2\right]$.

We believe that the proof that there are precisely 880 essentially distinct classic 4×4 fully-magic squares was first given by Fitting [5], though in [6] Bernard Frénicle de Bessy (c. 1605–1675) enumerated and classified these 880 matrices over 200 years earlier. Fitting [5] also showed that precisely 528 of these 880 have all four binary component matrices $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$ fully-magic, while very recently Amela [1] and Setsuda [7] have shown that 128 more, and so precisely 656 of these 880 have both Euler component matrices $\mathbf{L}_1$, $\mathbf{L}_2$ fully-magic. Brigadier-General F.J. Anderson (1860–1920) observed in [2] that certain 4×4 classic magic matrices have symmetric graphs.

In this talk we present a new and interesting classification of these 880 matrices using the 5 Frénicle–Amela patterns [1, 6], the 12 Dudeney types [3], and the Anderson symmetric graph property [2]. We have tried to illustrate our findings as much as possible, and whenever feasible with images of postage stamps or other philatelic items, with special emphasis on those associated with Leonhard Euler (1707–1783), Friedrich Fitting (1862–1945), Hans Fitting (1906–1938), and Brigadier-General Sir Francis James Anderson CB KBE (1860–1920).

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